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Multifractal Percolation and Growth in Intermittent Media

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The Havlin–Bunde multifractal hypothesis for the probability density of a random walker is used to obtain the scaling law of the pth-order correlation function of the concentration (for percolation) and of the height (for growing surfaces) differences: $c_p(r) = \langle |\Theta(x+r) - \Theta(x)|^p \rangle \sim r^{\zeta_p}$ in intermittent media. It is shown that near the transition to homogeneity $\zeta_p = Ap \ln(p/p_0)$ (where A and p_0 are some constants). Good agreement with recent experiments and computer simulations of different authors is established.

KEY WORDS: Scaling; percolation; growth; intermittency; moments.

1. In ref. 1 the multifractal hypothesis

$$\langle P^q \rangle \sim \langle P \rangle^{q^\gamma} \tag{1}$$

was formulated, where $P(\mathbf{x}, t)$ is the probability to find a random walker at time t at distance $|\mathbf{x}|$ from its starting point, $\gamma < 1$. This statement was rigorously proved in ref. 1 for linear fractals and it is strongly supported for percolation systems by numerical simulations.

On the other hand, it is shown in ref. 2 that the percolation process can be realized in such strong intermittent media as turbulence. In such media the Havlin–Bunde hypothesis can be extended (for large q) on the space differences

$$\Delta P(r) = |P(\mathbf{x} + \mathbf{r}) - P(\mathbf{x})| \tag{2}$$

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Indeed, if we consider a sphere of radius r (with center at point $\mathbf{x}$) and take a set with finite number N of points on the sphere, then (under some general conditions)

$$\langle(\Delta P(r))^p\rangle \simeq \frac{\sum_{i=1}^{N}(\Delta P_i)^p}{N} \tag{3}$$

[here ΔP_i is value of $\Delta P(r)$ in the ith point of the set]. In the case of large p the main contribution to the sum (3) is given by *extremely* large (on the sphere) values of ΔP_i. In strong intermittent media one can expect $\Delta P_i \gg P(\mathbf{x})$ (with probability close to 1). Then, in strong intermittent media

$$\langle(\Delta P(r))^p\rangle \simeq \langle(P(\mathbf{x}+\mathbf{r}))^p\rangle$$

for the large p. One can see, however that the dependence on $\mathbf{x}$ appears on the right-hand side of this estimate. As shown in ref. 3, fully developed turbulence can be considered as quasihomogeneous. In terms of the Havlin–Bunde hypothesis this means that we should consider $\gamma \to 1$.[1] In this case we are dealing with a quasihomogeneous medium and consequently

$$\lim_{\gamma\to 1}\langle(\Delta P(r))^p\rangle \simeq \lim_{\gamma\to 1}\langle(P(r))^p\rangle \tag{4}$$

2. For our purposes it is suitable rewrite (1) in the dimensionless form

$$F_{np} \sim F_{nq}^{\rho_{npq}} \tag{5}$$

where

$$F_{np} = \frac{\langle P^p\rangle}{\langle P^n\rangle^{p/n}} \sim \langle P\rangle^{p^\gamma - (p/n)n^\gamma} \tag{6}$$

and

$$\rho_{npq} = \frac{p^\gamma - (p/n)\,n^\gamma}{q^\gamma - (q/n)\,n^\gamma} \tag{7}$$

Let us introduce an analogous dimensionless form for the differences

$$F^*_{np} = \frac{\langle(\Delta P_r)^p\rangle}{\langle(\Delta P_r)^n\rangle^{p/n}} \tag{8}$$

Then from (4)

$$\lim_{\gamma \to 1} F^*_{np} \sim \lim_{\gamma \to 1} (F^*_{nq})^{\rho_{npq}} \tag{9}$$

If there is scaling

$$\langle (\Delta P(r))^p \rangle \sim r^{\zeta_p} \tag{10}$$

Then from (9) and (7) we obtain

$$\lim_{\gamma \to 1} \frac{\zeta_p - (p/n)\,\zeta_n}{\zeta_q - (q/n)\,\zeta_n} = \lim_{\gamma \to 1} \frac{p^\gamma - (p/n)\,n^\gamma}{q^\gamma - (q/n)\,n^\gamma} \tag{11}$$

Since

$$\lim_{\gamma \to 1} \rho_{npq} = \frac{p}{q} \frac{\ln(p/n)}{\ln(q/n)} \tag{12}$$

we obtain for ζ_p the functional equation

$$\frac{\zeta_p - (p/n)\,\zeta_n}{\zeta_q - (q/n)\,\zeta_n} = \frac{p}{q} \frac{\ln(p/n)}{\ln(q/n)} \tag{13}$$

3. It is easy to show that general solution of the functional equation (13) is

$$\zeta_p = Ap \ln(p/p_0) \tag{14}$$

where A and p_0 are some constants. To compare this result with the experimental data it is suitable rewrite (14) in the form

$$\frac{\zeta_p}{p} = a + b \ln p \tag{15}$$

If we choose the axes or coordinates (y, x) so that, $y = \zeta_p/p$ and $x = \ln p$, then Eq. (15) [and (14)] is represented by a *straight* line.

Since $P(r, t)$ can be considered as the concentration of a passive scalar, we can use recent experimental data on the multifractality of passive scalar differences in turbulent media. Figure 1 shows the experimental data obtained in the atmosphere (25 m above the ground).[4] The passive scalar in this experiment was temperature.[3] The straight line is drawn in the Fig. 1 for comparison with Eq. (15). The same values of ζ_p were also

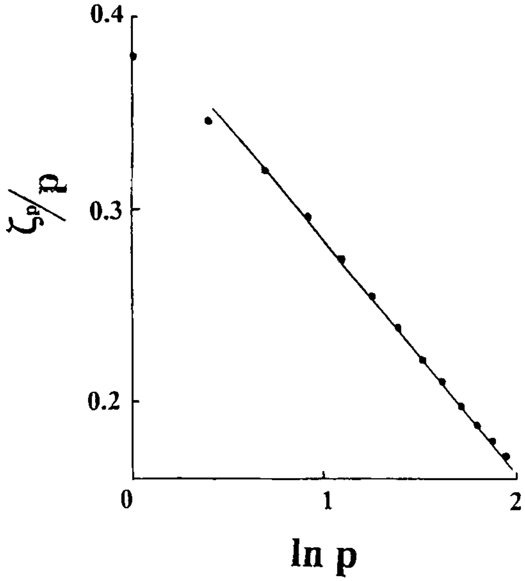

Fig. 1. The scaling exponent of the concentration difference moments obtained in the atmosphere.[4] The straight line corresponds to Eq. (15). The same data (for integer values of p) were also obtained in the experiment of ref. 5.

obtained in another recent experiment,[5] which can be an indication that the Havlin–Bunde hypothesis is valid for turbulent percolation.

4. It seems natural to apply analogous considerations to kinetic surface roughening with power-law-distributed amplitudes of uncorrelated noise.[6,7] The appropriately normalized qth-order correlation function of the height differences

$$c_p(r) = \langle |h(x+r) - h(x)|^p \rangle \sim r^{\zeta_p} \tag{16}$$

should be used in this case instead of (10).[6,7] Figure 2 (adapted from ref. 6) shows the results of a recent large-scale simulations of kinetic surface roughening with power-law-distributed amplitudes of uncorrelated noise. Already the authors of ref. 6 pointed out that the sharp change at $p \sim 3$ ($\ln p \sim 1$) can be an indication of a phase transition (in our terms this is the phase transition form random fractality to homogeneity: $\gamma \to 1$). The straight line in Fig. 2 is drawn for comparison with (15) (cf. Fig. 1).

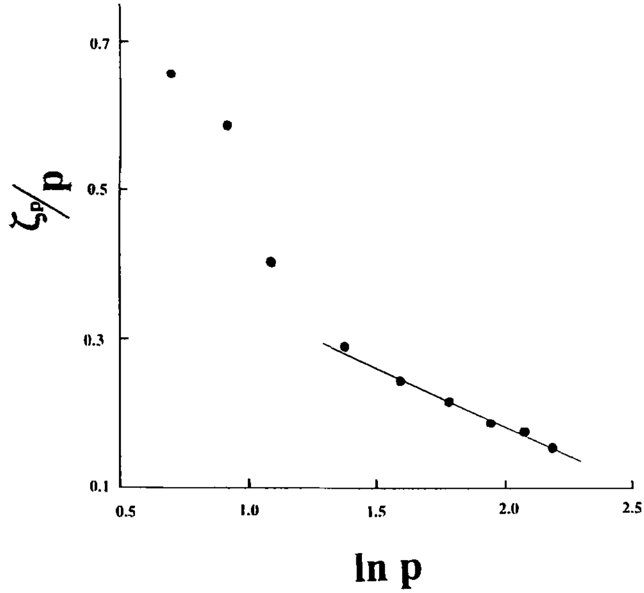

Fig. 2. The scaling exponent of the pth order correlation function of the height differences obtained in large-scale computer simulations of kinetic surface roughening with power-law-distributed amplitudes of uncorrelated noise (adapted from ref. 6). The straight lines corresponds to Eq. (15).

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